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COMMENT

Application of nonlinear deformation algebra to a physical system with Pöschl–Teller potential

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Abstract. We comment on a recent paper by Chen *et al* (1998 *J. Phys. A: Math. Gen.* **31** 6473), wherein a nonlinear deformation of $su(1,1)$ involving two deforming functions is realized in the exactly solvable quantum-mechanical problem with Pöschl–Teller potential, and is used to derive the well known $su(1,1)$ spectrum-generating algebra of this problem. We show that one of the defining relations of the nonlinear algebra, presented by the authors, is only valid in the limiting case of an infinite square well, and we determine the correct relation in the general case. We also use it to establish the correct link with $su(1,1)$, as well as to provide an algebraic derivation of the eigenfunction normalization constant.

In an interesting paper (henceforth referred to as I and whose equations will be quoted by their number preceded by I), Chen *et al* [1] recently pointed out that the nonlinear deformations of the $su(2)$ and $su(1,1)$ Lie algebras with two deforming functions $f(J_0)$ and $g(J_0)$, introduced by Delbecq and Quesne [2], can find some useful applications in quantum mechanics. They indeed claim to have proved that one of such algebras can be realized in a physical system with Pöschl–Teller potential, which is one of the exactly solvable one-dimensional quantum-mechanical potentials.

By starting from the 'natural' quantum operators X, P of Nieto and Simmons [3], they constructed mutually adjoint lowering and raising operators b, b^+, which together with the Hamiltonian H generate a nonlinear algebra with two deforming functions $f(H)$ and $g(H)$. They also obtained the eigenvalues and (unnormalized) eigenfunctions of H by using this algebra instead of solving the Schrödinger equation, and pointed out a relation with the well known $su(1,1)$ symmetry of the Pöschl–Teller potential (see [4] and references quoted therein).

In the present comment, we want to show that one of the defining relations of the nonlinear algebra, as given in I, is not entirely correct, and should actually contain an additional term, which only disappears in the $\nu \to 1$ limit, corresponding to an infinite square well. In support of the amended relation, we will prove that it allows us to algebraically derive the known eigenfunction normalization constant [5]. Finally, we will establish the correct relation between the nonlinear algebra and $su(1,1)$.

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Let H, b, b^+ be defined as in I by

$$H = \frac{p^2}{2m} + V(x) \qquad V(x) = \frac{V_0}{\cos^2(kx)} \qquad V_0 = \epsilon\nu(\nu-1) \qquad \epsilon = \frac{\hbar^2 k^2}{2m} \tag{1}$$

$$b = \frac{1}{2\epsilon}\left[X\left(\epsilon + 2\sqrt{\epsilon H}\right) + \frac{\mathrm{i}\hbar}{m}P\right] \tag{2}$$

$$b^+ = (b)^\dagger = \frac{1}{2\epsilon}\left[\left(\epsilon + 2\sqrt{\epsilon H}\right)X - \frac{\mathrm{i}\hbar}{m}P\right] = -\frac{1}{2\epsilon}\left[X\left(\epsilon - 2\sqrt{\epsilon H}\right) + \frac{\mathrm{i}\hbar}{m}P\right]\frac{\epsilon + \sqrt{\epsilon H}}{\sqrt{\epsilon H}} \tag{3}$$

where

$$X = \sin(kx) \qquad P = \tfrac{1}{2}k\{\cos(kx), p\} = k[\cos(kx)p + \tfrac{1}{2}\mathrm{i}\hbar k\sin(kx)] \tag{4}$$

satisfy the commutation relations

$$[X, P] = \mathrm{i}\hbar k^2(1 - X^2) \qquad [H, X] = -\frac{\mathrm{i}\hbar}{m}P$$
$$[H, P] = \mathrm{i}\hbar k^2\left(2XH - \frac{1}{2}\epsilon X - \frac{\mathrm{i}\hbar}{m}P\right). \tag{5}$$

Note that in equations (2) and (3), we have set $\gamma = (2\epsilon)^{-1}$ in accordance with equation (I45), and that ν can be expressed in terms of V_0 as

$$\nu = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4V_0}{\epsilon}}\right). \tag{6}$$

Equation (6) differs from equation (I35), wherein a minus sign is used before the square root. The plus sign is actually imposed by the condition that the wavefunctions vanish at the boundaries of the interval $(-\frac{\pi}{2k}, \frac{\pi}{2k})$.

For the commutators of H, b, and b^+, we obtain the results

$$[H, b] = -bg(H) \qquad [H, b^+] = g(H)b^+ \qquad g(H) = -\epsilon + 2\sqrt{\epsilon H} \tag{7}$$

$$[b, b^+] = -f(H) = 1 + 2\sqrt{H/\epsilon} + \frac{\nu(\nu-1)}{\sqrt{H/\epsilon}\left(\sqrt{H/\epsilon} - 1\right)} \tag{8}$$

which only partly agree with equations (I17) and (I18), as the last term on the right-hand side of equation (8) is missing there. Since the calculation of the commutator of b with b^+ is not quite straightforward, we shall now provide some steps of the proof of equation (8).

By using equations (2) and (3), as well as the method devised in I for commuting a function of H with X or P, we get

$$bb^+ = -\frac{1}{4\epsilon^2}\left\{\left[X^2\left(\epsilon - 2\sqrt{\epsilon H}\right) + \frac{\mathrm{i}\hbar}{m}XP\right]\left(3\epsilon + 2\sqrt{\epsilon H}\right) + \frac{\mathrm{i}\hbar}{m}PX\left(\epsilon - 2\sqrt{\epsilon H}\right) - \frac{\hbar^2}{m^2}P^2\right\}\frac{\epsilon + \sqrt{\epsilon H}}{\sqrt{\epsilon H}} \tag{9}$$

$$b^+b = \frac{1}{4\epsilon^2}\left\{\left[X^2\left(\epsilon + 2\sqrt{\epsilon H}\right) + \frac{\mathrm{i}\hbar}{m}XP\right]\left(3\epsilon - 2\sqrt{\epsilon H}\right) + \frac{\mathrm{i}\hbar}{m}PX\left(\epsilon + 2\sqrt{\epsilon H}\right) - \frac{\hbar^2}{m^2}P^2\right\}\frac{\sqrt{\epsilon H}}{\epsilon - \sqrt{\epsilon H}}. \tag{10}$$

From these results and equation (5), we obtain the rather complicated expression

$$[b, b^+] = \frac{1}{4\epsilon}\left\{-2[\epsilon^3 - 2\epsilon^2(\epsilon H)^{1/2} + 4(\epsilon H)^{3/2}] - X^2\epsilon(\epsilon^2 - 4\epsilon H) - 4\frac{\mathrm{i}\hbar}{m}\epsilon^2 XP + \frac{\hbar^2}{m^2}\epsilon P^2\right\} \times \frac{1}{\sqrt{\epsilon H}\left(\epsilon - \sqrt{\epsilon H}\right)}. \tag{11}$$

The latter can, however, be simplified by noting that

$$\nu(\nu-1) = \frac{1}{\epsilon}(1-X^2)\left(H - \frac{p^2}{2m}\right)$$
$$= \frac{1}{4\epsilon^2}\left[2(\epsilon^2+2\epsilon H) + X^2(\epsilon^2-4\epsilon H) + 4\frac{\mathrm{i}\hbar}{m}\epsilon X P - \frac{\hbar^2}{m^2}P^2\right] \qquad (12)$$

where the first equality directly results from equation (1). To obtain the second equality, use has been made of equation (4), leading to

$$[\cos(kx)p]^2 = \left(-\frac{1}{2}\mathrm{i}\hbar k X + \frac{P}{k}\right)^2 \qquad (13)$$

or

$$(1-X^2)p^2 + \frac{1}{2}\hbar^2k^2X^2 + \mathrm{i}\hbar X P = -\frac{1}{2}\hbar^2k^2 + \frac{1}{4}\hbar^2k^2X^2 - \mathrm{i}\hbar X P + \frac{1}{k^2}P^2. \qquad (14)$$

Inserting equation (12) into equation (11) completes the proof of equation (8).

We conclude that $J_0 = H$, $J_+ = b^+$, and $J_- = b$ do indeed define a nonlinear algebra with two deforming functions $f(H)$ and $g(H)$, as introduced in [2], but that $f(H)$ does contain an extra term not obtained in I. Since such a term depends upon the potential strength, we get different algebras for different Hamiltonians. It should be noted that the presence of $\sqrt{H/\epsilon}$ in the denominator does not lead to any problem when acting on the Hamiltonian eigenstates.

The Casimir operator of the nonlinear algebra is given by [2]

$$C = bb^+ + h(H) = b^+b + h(H) - f(H) \qquad (15)$$

where $h(H)$ satisfies the relation $h(H) - h(H - g(H)) = f(H)$, and is given by

$$h(H) = -\left(1+\sqrt{H/\epsilon}\right)^2 + \frac{\nu(\nu-1)}{\sqrt{H/\epsilon}}. \qquad (16)$$

This result, too, differs from equations (I2) and (I19).

The nonlinear algebra can be used to determine the spectrum of H and to construct all its eigenstates, and is therefore a spectrum-generating algebra. As proved in I, the eigenvalues and the ground state wavefunction, obtained from the equation

$$b|\psi_0\rangle = 0 \qquad (17)$$

are given by

$$E_n = \epsilon(n+\nu)^2 \qquad n = 0, 1, 2, \ldots \qquad (18)$$

and

$$\psi_0(x) = \mathcal{N}_0 \cos^\nu(kx) \qquad (19)$$

respectively. The normalization constant $\mathcal{N}_0$, calculated by direct integration, is

$$\mathcal{N}_0 = \left(\frac{k\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma(\nu+1/2)}\right)^{1/2}. \qquad (20)$$

The excited states $|\psi_n\rangle$, $n = 1, 2, \ldots$, can be obtained by repeatedly using the relation

$$b^+|\psi_n\rangle = \alpha_{n+1}|\psi_{n+1}\rangle \qquad (21)$$

where α_{n+1} is some yet unknown constant, which we may assume real and non-negative. By using the explicit form of b^+ and the Hamiltonian eigenvalues, given in equations (3) and (18), respectively, we get the recursion relation

$$\frac{n+\nu+1}{n+\nu}\left[-\frac{1}{k}\cos(kx)\frac{\mathrm{d}}{\mathrm{d}x} + (n+\nu)\sin(kx)\right]\psi_n(x) = \alpha_{n+1}\psi_{n+1}(x). \qquad (22)$$

If we set

$$\psi_n(x) = (1 - X^2)^{\nu/2}\phi_n(X) \tag{23}$$

where X is defined in equation (4), then equation (22) becomes

$$\left[(X^2 - 1)\frac{\mathrm{d}}{\mathrm{d}X} + (n + 2\nu)X\right]\phi_n(X) = \frac{n+\nu}{n+\nu+1}\alpha_{n+1}\phi_{n+1}(X). \tag{24}$$

Comparison with the differential and recursion relations of Gegenbauer polynomials [6] shows that

$$\phi_n(X) = \mathcal{N}_n C_n^{(\nu)}(X) \tag{25}$$

where the normalization constant $\mathcal{N}_n$ satisfies the recursion relation

$$\frac{\mathcal{N}_n}{\mathcal{N}_{n-1}} = \frac{n(n+\nu)}{(n+\nu-1)\alpha_n}. \tag{26}$$

Equations (23) and (25), which were already obtained before by Eleonsky and Korolev [7], are equivalent to equations (I39) and (I40), but provide a simpler expression for the wavefunctions.

To get the wavefunction normalization constant $\mathcal{N}_n$, it is clear from equation (26) that we need an explicit expression for α_n. By considering the diagonal matrix element (with respect to $|\psi_n\rangle$) of both sides of equation (8), and using equation (18), as well as the relation

$$b|\psi_n\rangle = \alpha_n|\psi_{n-1}\rangle \tag{27}$$

we obtain the following recursion relation for $|\alpha_n|^2$:

$$|\alpha_n|^2 - |\alpha_{n-1}|^2 = 2n + 2\nu - 1 + \frac{\nu(\nu-1)}{(n+\nu-1)(n+\nu-2)}. \tag{28}$$

Its solution is given by

$$|\alpha_n|^2 = (n+\nu)^2 - \frac{\nu(\nu-1)}{n+\nu-1} + \beta \tag{29}$$

where β is some constant. Since equation (17) imposes the condition $\alpha_0 = 0$, we get $\beta = -\nu(\nu-1)$. Hence

$$\alpha_n = \left(\frac{n(n+\nu)(n+2\nu-1)}{n+\nu-1}\right)^{1/2} \tag{30}$$

in accordance with equations (3.30) and (3.31) of [3].

Inserting equation (30) into equation (26) leads to the result

$$\mathcal{N}_n = \mathcal{N}_0\left(\frac{n!(n+\nu)\Gamma(2\nu)}{\nu\Gamma(n+2\nu)}\right)^{1/2}. \tag{31}$$

By combining equations (20), (23), (25), and (31), we obtain the following form for the normalized wavefunctions:

$$\psi_n(x) = \left(\frac{k(n!)(n+\nu)\Gamma(\nu)\Gamma(2\nu)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})\Gamma(n+2\nu)}\right)^{1/2}\cos^\nu(kx)C_n^{(\nu)}(\sin(kx)). \tag{32}$$

It only remains to take the known relation between Gegenbauer polynomials and associated Legendre functions [6] into account to get the equivalent form

$$\psi_n(x) = \left(\frac{k(n+\nu)\Gamma(n+2\nu)}{n!}\right)^{1/2}\cos^{1/2}(kx)P^{1/2-\nu}_{n+\nu-1/2}(\sin(kx)) \tag{33}$$

given by Nieto [5].

Equation (30), together with equations (18), (21), and (27), also allows us to determine the eigenvalue of the Casimir operator (15) corresponding to $|\psi_n\rangle$,

$$C|\psi_n\rangle = -\nu(\nu - 1)|\psi_n\rangle. \tag{34}$$

Hence, all the Hamiltonian eigenstates $|\psi_n\rangle$ belong to a single unitary irreducible representation of the nonlinear algebra, which may be characterized by ν.

At this stage, we may transform the nonlinear algebra in two different ways: either by trying to free ourselves from the need for considering different algebras for different Hamiltonians, or by restricting ourselves to the irreducible representation wherein equation (34) is satisfied.

In the former case, we may use equations (15) and (16) to express $\nu(\nu - 1)$ in terms of C, bb^+, and $\sqrt{H/\epsilon}$. Inserting such an expression into equation (8) then leads to an extended nonlinear algebra, generated by H, b, b^+, C, and characterized by the defining relations (7), as well as

$$[C, H] = [C, b] = [C, b^+] = 0 \tag{35}$$

$$\sqrt{H/\epsilon}\,bb^+ - \left(\sqrt{H/\epsilon} - 1\right) b^+b = C + \sqrt{H/\epsilon}\left(1 + 3\sqrt{H/\epsilon}\right). \tag{36}$$

Such an algebra may serve as a spectrum-generating algebra for the whole class of Pöschl–Teller Hamiltonians (1).

In the latter case, we may replace $\nu(\nu - 1)$ by $-C$ in equation (8). Doing the same in equations (15) and (16), we may express C in terms of bb^+ and $\sqrt{H/\epsilon}$ as

$$C = \frac{\sqrt{H/\epsilon}}{\sqrt{H/\epsilon} + 1} bb^+ - \sqrt{H/\epsilon}\left(\sqrt{H/\epsilon} + 1\right). \tag{37}$$

Inserting this expression into the transformed equation (8) leads to the relation

$$\frac{\sqrt{H/\epsilon}}{\sqrt{H/\epsilon} + 1} bb^+ - \frac{\sqrt{H/\epsilon} - 1}{\sqrt{H/\epsilon}} b^+b = 2\sqrt{H/\epsilon}. \tag{38}$$

Hence, the operators

$$\begin{aligned} J_0 &= \sqrt{H/\epsilon} \qquad J_+ = b^+\left(\frac{\sqrt{H/\epsilon}}{\sqrt{H/\epsilon} + 1}\right)^{1/2} = \left(\frac{\sqrt{H/\epsilon} - 1}{\sqrt{H/\epsilon}}\right)^{1/2} b^+ \\ J_- &= \left(\frac{\sqrt{H/\epsilon}}{\sqrt{H/\epsilon} + 1}\right)^{1/2} b = b\left(\frac{\sqrt{H/\epsilon} - 1}{\sqrt{H/\epsilon}}\right)^{1/2} \end{aligned} \tag{39}$$

satisfy the defining relations of $su(1, 1)$

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = -2J_0 \tag{40}$$

while the operator (37) reduces to the $su(1, 1)$ Casimir operator, $C = J_-J_+ - J_0(J_0 + 1)$. We conclude that under the substitution of $-C$ for $\nu(\nu - 1)$, the spectrum-generating nonlinear algebra becomes equivalent to the well known $su(1, 1)$ algebra of the Pöschl–Teller potential.

As a final point, let us comment on the limit $V_0 \to 0$ or $\nu \to 1$, corresponding to an infinite square well of width $L = \pi/k$. In such a case, the additional term in equation (8) vanishes, so that the results of I are applicable. In particular, equation (I48) provides an acceptable realization of $su(1, 1)$. Since such a realization differs from equation (39), one may wonder at such a discrepancy. The latter is, however, easily understood by noting that with realization (I48) the nonlinear algebra Casimir operator C actually differs from that of $su(1, 1)$ by an additive constant, whereas with realization (39) both exactly coincide.

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